Using Markov and semi-Markov Reward Systems to Assess Patient Costs

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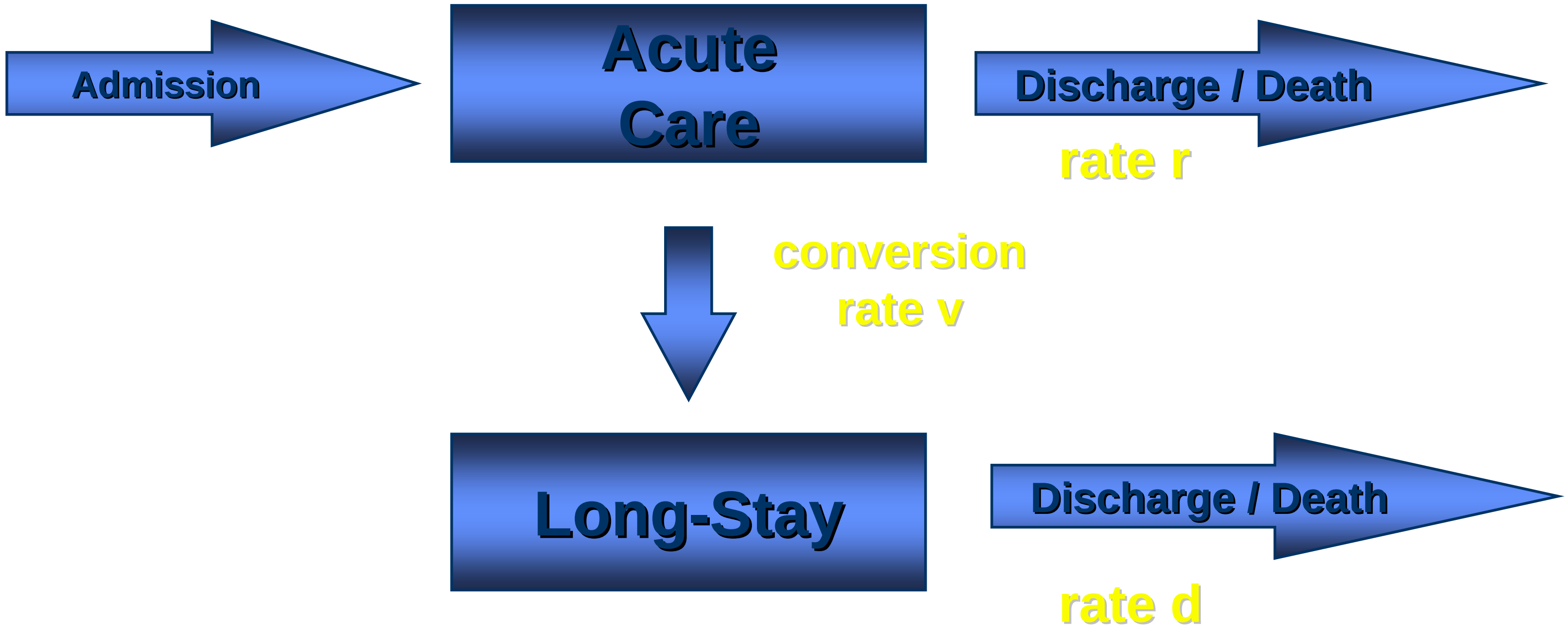
Modelling Patient Costs

- Users of medical services may occupy hospital beds for prolonged periods of time thus consuming scarce resources.
- It is therefore important that decision makers understand how clinical and social decisions interact to influence long-term care costs.
- A modelling approach allows us to assess different care options, thereby facilitating better decision making.

Phases of Patient Care

- Patients may be thought of as progressing through stages of acute care, rehabilitation and long-stay care where most patients are eventually rehabilitated and discharged.
- Such a situation occurs when there are a number of distinct methods of managing in-patients. Thus an acute phase may be relatively quick, lasting for days or possibly weeks.
- A long-stay phase, on the other hand, may involve patients remaining in hospital for months, or even years.
- These patients may be very consuming of resources and thereby distort the performance statistics and cost implications.

A Two Phase Model of Patient Flows



λ_i

A Markov Reward Model

- We define hospital care as a k phase Markov model with a k+1st absorbing state (typically death/discharge).
- The transition matrix of probabilities of movement from one state to another is given by $A = \{a_{ij}\}$.
- New patients are admitted to the i^{th} phase of hospital care according to a Poisson process, rate λ_ι
- There are initially ν_i patients in phase i.
- There is a cost c_i at each time point for each patient in phase i.

The mean cost

The mean cost at time *t* is given by:

$C(t) = \Sigma \nu_i \mathbf{V}_i \mathbf{A}^t \mathbf{c} + \Lambda \Sigma \mathbf{A}^{t-s} \mathbf{c}$

where V_i is a row vector with 1 in the ith position.

As $t \rightarrow \infty$ the limiting distribution of costs is therefore Poisson, with mean costs for each state given by:

$C_{\infty} = \lim \Lambda \Sigma \mathbf{A}^t \mathbf{c} = \Lambda(\mathbf{I}-\mathbf{A})^{-1}.\mathbf{c}$

An illustrative Example

- We consider two scenarios of the two-state model
  - The therapeutic model, has longer mean lengths of stay in each state but a smaller probability of becoming long-stay, while
  - the prosthetic model has shorter mean lengths of stay in each state but are less likely to progress from acute to long-term care.

Parameters	Therapeutic model	Prosthetic model
r	0.3	0.25
v	0.1	0.25
d	0.08	0.1
ν_1	50	50
ν_2	0	0
λ	20	20
c_1	1000	1000
c_2	500	500
mean1	2.5	2.0
mean2	12.5	10.0

Costs of the Therapeutic and Prosthetic models

Time (weeks)	therapeutic cost $c_1(t)$	prosthetic cost $c_2(t)$
0	70000	70000
26	78032	87375
52	80881	89830
78	81208	89999
104	81245	90000
130	81250	90000

The semi-Markov Reward System

We define states of a semi-Markov system as S_1, …S_k and S_{k+1} where S_{k+1} is an absorbing state. The transition matrix of probabilities of eventual first move from one grade to another is then given by A = $\{a_{ij}\}$ with the corresponding holding time probabilities,

Then, we define the probability of transition as:

$P_{ij}(t)$ =Probability{in S_j at time t | in S_i at time zero}, Let $R_i(t)$ be the total number of individuals recruited at time t to grade S_i, where

R_i (t)~Poisson(λ_i for i =1,…,k.

Let ν_i be the number of individuals in grade S_i at time t=0 for i=1, …, k.

Notation

We define $\mathbf{R}(t) = \sum_{s=0}^{t} \mathbf{P}(t-s)$ and $\mathbf{Z} = \{Z_i\}$, $Z_i = 1 - z_i$, $\mathbf{P}(t) = \{P_{ij}(t)\}$, $\Lambda = \{\lambda_i\}$ and $\mathbf{V}_i = (0,0\ldots,1,\ldots0)$ is a row vector with 1 in the i th position.

We associate a reward c_i with each individual at each time point in state S_i. The joint p.g.f. of the total cost, for each state, at time *t*, is then given by:

Mean costs for the semi-Markov model

$$\boldsymbol{\Gamma}(\mathbf{Z},t)=\prod_{i=1}^{k}(1+\mathbf{V}_i\mathbf{P}(t)\mathbf{Z_c})^{\nu_i}.\exp\{\boldsymbol{\Lambda}\mathbf{R}(t)\mathbf{Z_c}\}$$

where $\mathbf{Z_c}=\{1-z_i^{c_i}\}$.

The mean cost at time t is then:

$$\mathbf{C}(t)=\sum_{i=1}^{k}\nu_i\mathbf{V}_i\mathbf{P}(t)\mathbf{c}+\boldsymbol{\Lambda}\mathbf{R}(t)\mathbf{c}$$

$$=\sum_{i=1}^{k}\nu_i\sum_{j=1}^{k}c_jP_{ij}(t)+\sum_{i=1}^{k}\lambda_i\sum_{j=1}^{k}R_{ij}(t)c_j.$$

A more detailed discussion and proof of these results is provided in McClean, Papadopolou and Tsaklides (2004).

The limiting distribution of costs

The limiting distribution of costs is therefore Poisson, with mean costs for each state given by:

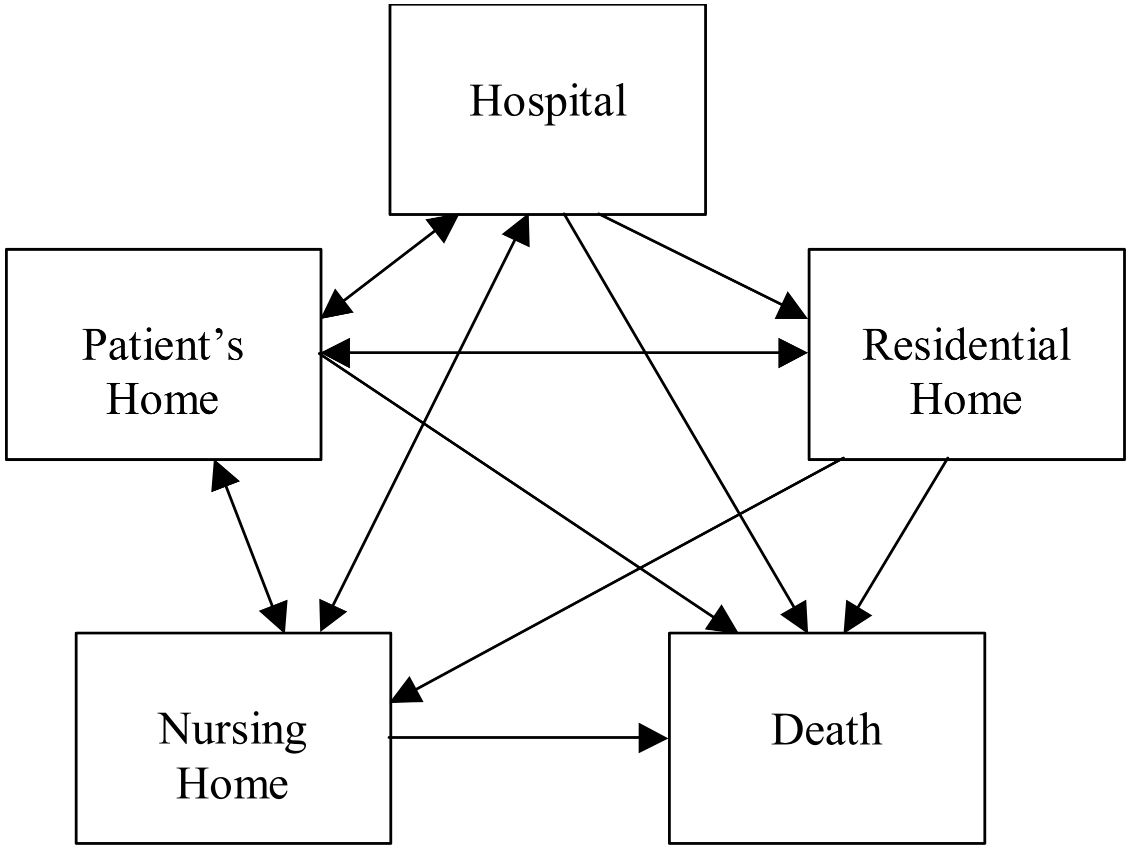
$$\mathbf{C}_\infty = \lim_{t\to\infty} \Lambda\mathbf{R}(t)\mathbf{c} = \Lambda\mathbf{\Pi}(1)\mathbf{c}$$

$$= \Lambda(\mathbf{I}-\mathbf{A})^{-1}\mathbf{D}.\mathbf{c}$$

where $\mathbf{D} = \mathrm{diag}\{\sum_{\substack{\mathrm{u=1} \\ \mathrm{u \neq i}}}^{\mathrm{k+1}} a_{iu}\overline{\tau}_{iu}\}$

and $\overline{\tau}_{iu}$ is the mean duration of the transition from S_i to S_u.

The Semi-Markov Geriatric System



Applying the model

The transition matrix is then:

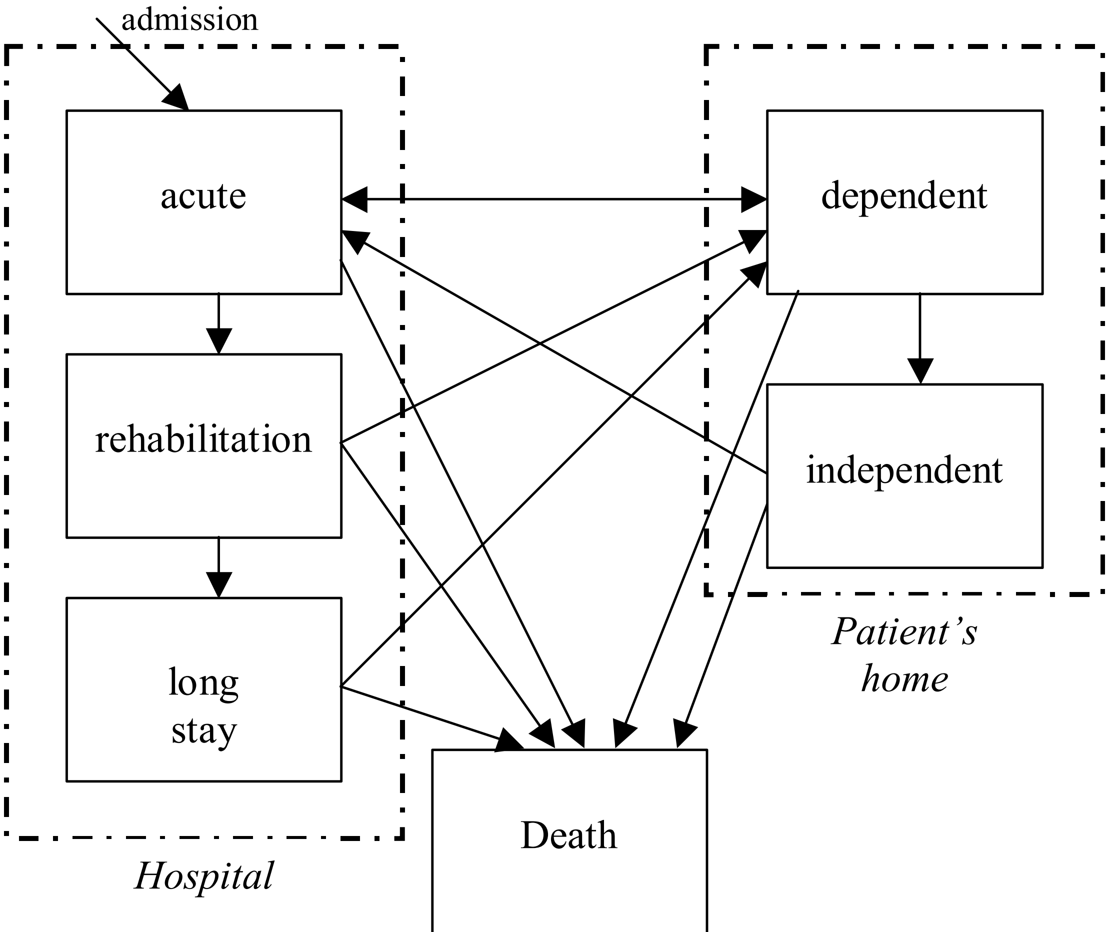
$$\mathbf{A} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}$$

We can determine the mean cost of the system with Poisson admissions at any time, and in steady state, in terms of the daily costs in each state and the mean duration times of the transitions between states.

The semi-Markov Reward System as a Virtual Markov System

- By utilising a phase-type model to describe durations in each state of the geriatric care system we may transform the semi-Markov reward system into a virtual Markov Reward system with latent states –the phases.
- Phase type models have previously been successfully fitted to data for length of stay of elderly people in hospital and social services care
- The approach is somewhat similar to the method of stages used for Erlang arrival and service distributions in Queueing Theory.

An Example


admission
acute
rehabilitation
long stay
Hospital
dependent
independent
Patient's home
Death

Conclusion

- We have developed a Markov and semi-Markov reward model for a k-state hospital system with Poisson admissions and an absorbing state, typically death or discharge.
- Average cost at any time is evaluated for a two state Markov system (acute/rehabilitative and long-stay) and two scenarios : the Therapeutic and Prosthetic models respectively.
- This example is used to illustrate the notion that keeping patients longer in hospital, and using rehabilitation to reduce the transfer of patients into long-stay care, may reduce costs.
- We extend our approach to a semi-Markov reward model,since durations are not usually exponential. However we can represent the semi-Markov model as a virtual Markov system.
- This allow us to explore financial implications for hospital management.

Further work

- Previously we have incorporated covariates into the phase-type distribution by allowing the parameters to depend on them through log-linear functions – this approach can be extended to the semi-Markov reward model.
- By assigning costs to the various states of the model and taking account of relevant covariates, we may thus determine the overall costs involved in treating groups of patients and assess various treatment regimes.
- Such approaches facilitate a systems approach to health care planning, thus enabling cost-effective strategies to be developed and assessed.